3rd Meeting of the GPD of the EPS
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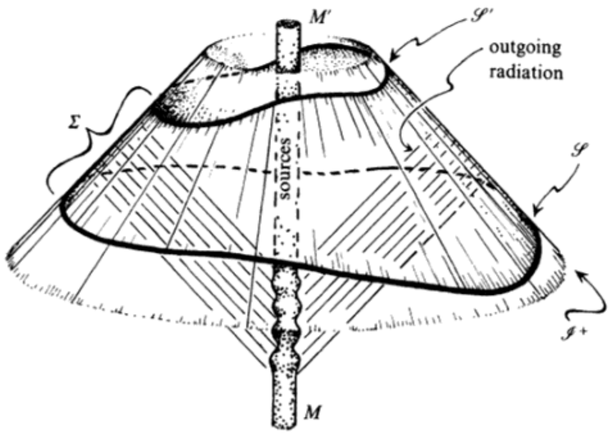
Asymptotic symmetries and conserved charges in gravity



Penrose & Rindler Vol II

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Noether's theorems revisited

Global symmetries $\quad \delta_Q \phi^i = Q^i[\phi, x] \;,\quad \delta_Q L = \partial_\mu k^\mu_Q$

$$Q^i \frac{\delta L}{\delta \phi^i} = \partial_\mu \big(\underbrace{k^\mu_Q - \frac{\partial L}{\partial \partial_\mu \phi^i} Q^i + \dots}\big)$$

constructive formula

$\Rightarrow$ conserved current $\quad \partial_\mu \; j^\mu_Q \approx 0$ $\quad$ Noether's 1st th.

Gauge symmetries $\quad \delta_\epsilon \phi^i = R^i_\alpha(\epsilon^\alpha) = R^i_\alpha \epsilon^\alpha + R^{i\mu}_\alpha \partial_\mu \epsilon^\alpha + \dots \;,\quad \delta_\epsilon L = \partial_\mu k^\mu_\epsilon \;,\quad \forall \epsilon^\alpha(x)$

$$\underset{\partial_\mu j^\mu_\epsilon}{\underset{\shortparallel}{R^i_\alpha(\epsilon^\alpha) \frac{\delta L}{\delta \phi^i}}} = \epsilon^\alpha \big[\underbrace{R^i_\alpha \frac{\delta L}{\delta \phi^i} - \partial_\mu \big(R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i}\big) + \dots}_{R^{+i}_\alpha \left(\frac{\delta L}{\delta \phi^i}\right)} \big] + \partial_\mu \big[\underbrace{\epsilon^\alpha R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i} + \dots}_{S^\mu_\epsilon} \big]$$

Noether's 2nd th.

$\forall \epsilon^\alpha(x) : \quad R^{+i}_\alpha \left(\frac{\delta L}{\delta \phi^i} \right) = 0$ $\quad$ Noether identities $\quad$ & $\quad \partial_\mu (j^\mu_\epsilon - S^\mu_\epsilon) = 0$

Algebraic Poincaré lemma : $j^\mu_\epsilon = \underset{\approx 0}{S^\mu_\epsilon} + \partial_\nu k^{[\mu\nu]}_\epsilon$ trivial Noether current

undetermined

NB : $d_H = dx^\mu \left(\frac{\partial}{\partial x^\mu} + \partial_\mu \phi^i \frac{\partial}{\partial \phi^i} + \dots \right) \neq d = dx^\mu \frac{\partial}{\partial x^\mu}$

$$d_H \omega^p = 0 \Rightarrow \omega^p = d_H \eta^{p-1}, \quad \eta^{p-1} = \rho_H \omega^p \neq d\omega^p = 0 \Rightarrow \omega^p = d\eta^{p-1}, \quad \eta^{p-1} = \rho\omega^p$$

not the same homotopy operator $\rho_{(H)}$, locality !

Complete Noether theorem : $[Q^i] \longleftrightarrow [j^{n-1}]$

1-1 correspondence

$\delta_Q L = \partial_\mu k^\mu_Q$

$d_H j^{n-1} \approx 0$

$Q^i \sim Q^i + R^i_\alpha(f^\alpha)$ + "trivial gauge symmetries"

$f^\alpha = f^\alpha[\phi, x]$

$j^{n-1} \sim j^{n-1} + \underset{\approx 0}{t^{n-1}} + d_H \eta^{n-2}$

Algebra : $\delta_{Q_1} j^{n-1}_{Q_2} = j^{n-1}_{[Q_1, Q_2]}$ + trivial + (central) extension

Generalization: Classification problem $k^{n-2} = k^{[\mu\nu]} \frac{1}{2!} \frac{1}{(n-2)!} t_{\mu\nu\alpha_3 \dots \alpha_n} dx^{\alpha_3} \dots dx^{\alpha_n}$

$d_H k^{n-2} \approx 0 \quad , \quad k^{n-2} \sim k^{n-2} + \underset{\approx 0}{t^{n-2}} + d_H \eta^{n-3}$

non trivial conserved co-dimension 2-forms

Result: $[k^{n-2}] \longleftrightarrow \bar{f}^{\alpha}, \quad R^i_{\alpha}(\bar{f}^{\alpha}) = 0$ reducibility parameters

1-1 correspondence

Examples: YM: $\delta_\epsilon A^\alpha_\mu = D_\mu \epsilon^\alpha \qquad D_\mu \bar{\epsilon}^\alpha = 0 \quad \begin{cases} \bar{\epsilon}^\alpha = 0 & \text{semi-simple gauge group} \\ \epsilon = cte & U(1) \end{cases}$

metric gravity $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \qquad \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0 \Rightarrow \bar{\xi}^\mu = 0$

generic metric has no killing vectors

Linearized GR around solution $\bar{g}_{\mu\nu}$

$$S^{(2)}[h,\bar{g}] = \frac{1}{16\pi G}\int d^n x\, L^{EH}, \qquad L^{EH} = \sqrt{|g|}\,(R-2\Lambda), \qquad \delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$$

$\mathcal{L}_{\bar\xi}\, \bar{g}_{\mu\nu} = 0$ Kvf of background solution

$$k_{\bar\xi}[h,\bar{g}] = \frac{1}{2} h_{\mu\nu} \frac{\delta}{\delta \partial_\alpha g_{\mu\nu}} \frac{\partial S_{\bar\xi}}{\partial dx^\alpha} + \frac{2}{3}\partial_\sigma \Big[h_{\mu\nu} \frac{\partial}{\partial \partial_\alpha \partial_\sigma g_{\mu\nu}} S_{\bar\xi}\Big], \qquad S_{\bar\xi} = 2\frac{\delta L^{EH}}{\delta g_{\mu\nu}} g_{\nu\rho}\bar\xi^\rho$$

= MTW, Abbott-Deser, Wald, Anderson-Torre, ... complicated because 2nd order

Remarks: • 1st order theories $S = \int d^n x \,[a^\mu_i(\phi)\partial_\mu \phi^i - h(\phi)]$ at most linear in $\partial_\mu \phi^i$

$$k_{\bar f}[\varphi,\bar\phi] = \left(R^{i\mu}_{\alpha}\, \sigma^\nu_{ij}\, \varphi^j \bar{f}^\alpha\right) \frac{1}{2!}\frac{1}{(n-2)!}\epsilon_{\mu\nu\alpha_3\ldots\alpha_n} dx^{\alpha_3}\ldots dx^{\alpha_n}$$

$\sigma^\nu_{ij} = \partial_i a^\nu_j - \partial_j a^\nu_i$ presymplectic $(2,n-1)$ form

Cartan formulation $S^C[e_a{}^\mu, \Gamma^a{}_{b\mu}] = \frac{1}{16\pi G} \int d^n x\, |e| \left(R^{ab}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu - 2\Lambda\right)$

$$\begin{cases} \delta_{\xi,\omega} e_a{}^\mu = \mathcal{L}_\xi e_a{}^\mu + \omega_a{}^b e_b{}^\mu \\ \delta_{\xi,\omega} \Gamma^a{}_{b\mu} = \mathcal{L}_\xi \Gamma^a{}_{b\mu} - D_\mu \omega^a{}_b \end{cases}$$

Newman-Penrose formulation $\Gamma^a{}_{bc} = \Gamma^a{}_{b\mu} e^\mu{}_c$

- associated global symmetries to $\bar{f}^\alpha$: $R^i_\alpha(f^\alpha) \frac{\delta L}{\delta \phi^i} = \partial_\mu S^\mu_f$

expand to 2nd order in φ^i $R^{i(0)}_\alpha(f^\alpha)\left(\frac{\delta L}{\delta \phi^i}\right)^{(2)} + R^{i(1)}_\alpha(f^\alpha)\left(\frac{\delta L}{\delta \phi^i}\right)^{(1)} + R^{i(2)}_\alpha(f^\alpha)\left(\frac{\delta L}{\delta \phi^i}\right)^{0} = \partial_\mu S^{\mu(2)}_f$

$f^\alpha = \bar{f}^\alpha$, $\left(\frac{\delta L}{\delta \phi^i}\right)^{(1)} = \frac{\delta L^{(2)}}{\delta \varphi^i}$ $R^{i(1)}_\alpha(\bar{f}^\alpha) \frac{\delta L^{(2)}}{\delta \varphi^i} = \partial_\mu S^{\mu(2)}_{\bar{f}}$

$\delta_{\bar{f}} \varphi^i = R^{i(1)}_\alpha(\bar{f}^\alpha)$ global symmetry of linearized theory

$\bar{f}^\alpha \nearrow$ surface charge $Q^{n-2}_{\bar{f}} = \oint_{\Sigma^{n-2}} k_{\bar{f}}[\varphi, \bar{\phi}]$ linear Algebra $\delta_{\bar{f}_1} k^{n-2}_{\bar{f}_2} = k^{n-2}_{[\bar{f}_1, \bar{f}_2]}$ + trivial

$\bar{f}^\alpha \searrow$ standard Noether charge $Q^{n-1}_{\bar{f}} = \oint_{\Sigma^{n-1}} S^{\mu(2)}_{\bar{f}} \frac{1}{(n-1)!} \epsilon_{\mu\alpha_2\ldots\alpha_n} dx^{\alpha_2}\ldots dx^{\alpha_n}$ quadratic

Ex: linearized gravity, flat spacetime (Pauli-Fierz theory)

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{\eta}_{\mu\nu}$, $\bar{\xi}^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$ Poincaré algebra

- $Q^{n-2}_{\bar{\xi}}$: ADM surface charges Hamiltonian analysis : involve $h^{L,T}_{ij}$, $\pi^{ij}_{L,T}$
  measure properties of matter $T^{\mu\nu}$ sources
  cf Gauss law $\partial_i \pi^i = j^0$

- $Q^{n-1}_{\bar{\xi}}$: global Poincaré charges for spin 2 gauge fields
  Hamiltonian analysis : involve physical dof h^{TT}_{ij}, π^{ij}_{TT}

Asymptotic analysis $\longrightarrow$ mixed-up.

Asymptotic symmetries

main idea: asymptotic symmetries = residual gauge symmetries

BMS ansatz

metric formulation
$$g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & -\frac{V}{r} e^{-2\beta} & -U^B e^{-2\beta} \\ 0 & -U^A e^{-2\beta} & g^{AB} \end{pmatrix}$$

$u \qquad r \qquad x^A = \theta, \phi \ldots \left(4d: \ \zeta = \cot\frac{\theta}{2} e^{i\phi}, \ \bar{\zeta} \right)$

d gauge conditions: $g^{uu} = 0 = g^{uA}$, $\det g_{AB} = r^{2(d-2)} \det \bar{\gamma}_{AB}$, $\bar{\gamma}_{AB}\, dx^A dx^B = d^{d-2}\Omega$

metric on unit d-2 sphere

fall-off conditions: $\beta = O(1)$, $U^A = O(1)$, $\frac{V}{r} = -\frac{r^2}{\ell^2} + O(r^2)$

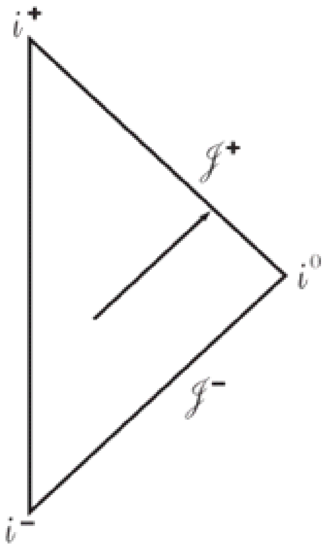
$$g_{AB}\, dx^A dx^B = r^2 \bar{\gamma}_{AB}\, dx^A dx^B + O(r^2)$$

originally motivated by gravitational wave solutions

Background solutions: $\beta = 0$, $U^A = 0$, $\frac{V}{r} = -\frac{r^2}{\ell^2} - 1$ $\quad d\bar{s}^2 = -(1+\frac{r^2}{\ell^2})du^2 - 2du\,dr + r^2 d^{d-2}\Omega$

$\ell^{-1} = 0$, $t = u + r$ Minkowski$_d$ | $\ell^{-1} \neq 0$, $t = u + \ell \arctan \frac{r}{\ell}$ AdS$_d$

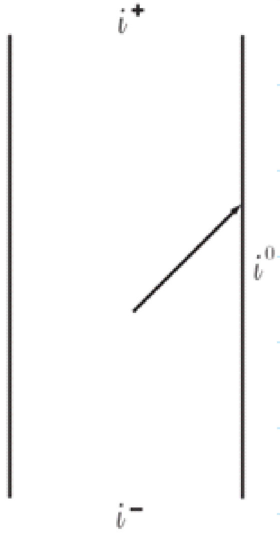
asymptotics: $r \to \infty$, u, x^A fixed



conf. rescaled metric

$d\tilde{s}^2 = 0\,du^2 + r^2 d^{d-2}\Omega$

null infinity



conf. rescaled metric

$d\tilde{s}^2 = -\ell^{-2} du^2 + d^{d-2}\Omega$

spatial infinity

Asymptotic symmetries: diffeo's that leave BMS ansatz invariant

Equations: $\mathcal{L}_\xi g^{uu} = 0 = \mathcal{L}_\xi g^{uA}$, $g^{AB}\mathcal{L}_\xi g_{AB} = 0$, $\mathcal{L}_\xi g^{ur} = O(1)$ ....

$\ell^{-1}=0$ (flat)

$d\geq 5$ $\quad so(d-1,1)\ltimes ST$

stronger fall-off's while keeping radiation $\quad iso(d-1,1)$

only exact Poincaré symmetries of background

$d=4$ $\quad so(3,1)\ltimes ST \qquad BMS_4$

not possible to remove supertranslations & keep radiation

$d=3$ $\quad diff(S^1)\ltimes C^\infty(S^1) \qquad BMS_3 \quad \supset \quad iso(2,1)$

$$\xi = Y\frac{\partial}{\partial\phi}+f\frac{\partial}{\partial u}, \quad f=T+uY', \quad Y=Y(\phi),\ T=T(\phi)$$

$$\begin{cases} i[j_m,j_n]=(m-n)j_{m+n}\\ i[j_m,p_n]=(m-n)p_{m+n}\\ i[p_m,p_n]=0\end{cases}$$

$\ell^{-1}\neq 0$ (Anti-de Sitter)

$d\geq 4$ $\quad so(d-1,2)$ (only exact Kvf of background)

$d=3$ $\quad diff(S^1)\oplus diff(S^1) \supset so(2,2)$

enhancement

$$\xi = Y^+\frac{\partial}{\partial x^+}+Y^-\frac{\partial}{\partial x^-} \qquad x^\pm=\frac{u}{\ell}\pm\phi$$

$$Y^+=Y^+(x^+),\quad Y^-=Y^-(x^-)$$

$$i[l^\pm_m,l^\pm_n]=(m-n)\,l^\pm_{m+n}$$

Current algebras

$d=3$ general solution Einstein equations $ds^2 = \left(-\frac{r^2}{\ell^2} + \mathcal{M}\right)du^2 - 2\,du\,dr + \mathcal{N}\,du\,d\phi + r^2\,d\phi^2$

$\ell^{-1}=0$ $\quad \mathcal{M} = \Theta,\ \mathcal{N} = \Xi + u\,\Theta'$
$\quad \Theta = \Theta(\phi),\ \Xi = \Xi(\phi)$

$\ell^{-1}\neq 0$ $\quad \mathcal{M}(u,\phi) = 2(T_{++} + T_{--}),\ \mathcal{N} = 2\ell(T_{++} - T_{--})$
$\quad T_{++} = T_{++}(x^+),\ T_{--} = T_{--}(x^-)$

residual symmetries : transform solutions to solutions

$$\begin{cases} -\delta_\xi \Theta = Y\Theta' + 2Y'\Theta - 2Y''' \\ -\delta_\xi \Xi = Y\Xi' + 2Y'\Xi + T\Theta' + 2T'\Theta - 2T''' \end{cases}$$

$$\begin{cases} -\delta_\xi T_{++} = Y^+\partial_+ T_{++} + 2\partial_+ Y^+ T_{++} - \frac{1}{2}\partial_+^3 Y^+ \\ -\delta_\xi T_{--} = Y^-\partial_- T_{--} + 2\partial_- Y^- T_{--} - \frac{1}{2}\partial_-^3 Y^- \end{cases}$$

currents : use k_ξ^{n-2} as in linearized theory

no guarantee for
- integrability ? $k_\xi^{n-2} = \delta J_\xi^{n-2}$ (large r) no problem in 3d, relevant in 4d
- conservation ? because ξ not exact Kof of background

current algebra $\quad \delta_{\xi_1} J^a_{\xi_2} = J^a_{[\xi_1,\xi_2]} + K^a_{\xi_1,\xi_2} + \text{trivial} \qquad x^a = (u,\phi)$

$\ell^{-1} = 0$ | currents | $\ell^{-1} \neq 0$

$$J^u_\xi = \frac{1}{16\pi G}\left[T \Theta + Y \Xi\right], \quad J^\phi_\xi = 0 \qquad\qquad J^\pm_\xi = \frac{\ell}{4\pi G} Y^\mp \Xi_{\mp\mp}$$

extensions

$$K^u_{\xi_1,\xi_2} = \frac{1}{16\pi G}\left[Y_1' T_2'' + T_1' Y_2'' - (1\leftrightarrow 2)\right], \quad K^\phi_{\xi_1,\xi_2} = 0 \qquad\qquad K^\pm = \frac{1}{16\pi G}\left[Y_1^{\mp\prime} Y_2^{\mp\prime\prime} - (1\leftrightarrow 2)\right]$$

charges

$$Q_\xi = \frac{1}{16\pi G}\oint_0^{2\pi} d\phi \, J^u_\xi \qquad\qquad Q_\xi = \frac{\ell}{8\pi G}\oint_0^{2\pi} d\phi \left(Y^+ \Xi_{++} + Y^- \Xi_{--}\right)$$

mode expansion

$$\begin{cases} i[J_m, J_n] = (m-n) J_{m+n} \\ i[J_m, P_n] = (m-n) P_{m+n} + \frac{c}{12}\delta^0_{m+n}\, m(m^2-1) \\ i[P_m, P_n] = 0 \end{cases} \qquad\qquad i[L^\pm_m, L^\pm_n] = (m-n) L^\pm_{m+n} + \frac{c^\pm}{12}\delta^0_{m+n}\, m(m^2-1)$$

central charges

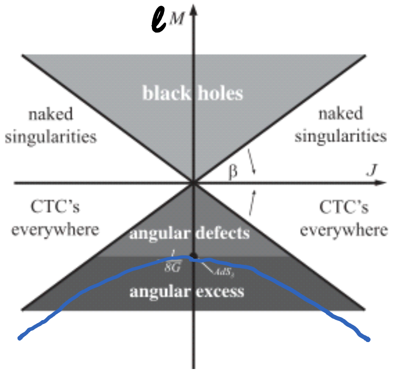
$$c = \frac{3}{G} \qquad\qquad c^\pm = \frac{3\ell}{2G}$$

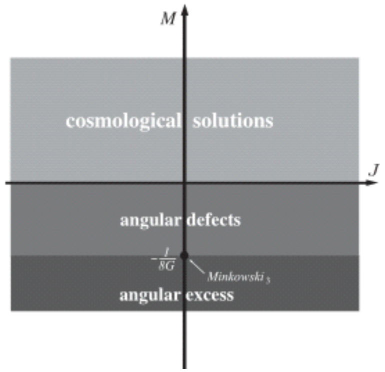
3d gravity   charges : $\langle \text{Sol}, \mathfrak{g} \rangle \longrightarrow \mathbb{R}$

Solution space = coadjoint representation of $\hat{\mathfrak{g}}^*$

classified into coadjoint orbits of $\left\{ \begin{array}{l} BMS_3 \\ \\ \text{Virasoro} \otimes \text{Virasoro} \end{array} \right.$

Simplest solutions constants $\Theta = M$, $\Xi = J$ $\Xi_{\substack{++\\--}} = 2G\left(M \pm \frac{J}{\ell}\right)$





$d=4$, $\ell^{-1}=0$ stereographic coordinates $\zeta = \cot\frac{\theta}{2}\, e^{-i\phi}$, $\bar\zeta$

$d\bar s^2 = -2(P_s\bar P_s)^{-1}\, d\zeta\, d\bar\zeta$ $\quad P_s = \frac{1}{R\sqrt{2}}(1+\zeta\bar\zeta)$ conformally flat

covariant derivative $D_A \leftrightarrow \eth\eta^{s,w} = P_s\bar P_s^{-s}\,\partial\,(\bar P_s^{s}\eta^{s,w})$ $\quad \bar\eth\eta^{s,w} = \bar P_s P_s^{s}\,\bar\partial(P^{-s}\eta^{s,w})$

$[s,w]$ spin & boost weights

bms$_4$ $\quad \mathcal{Y}^{-1,0}$, $\bar{\mathcal{Y}}^{1,0}$, $\mathcal{T}^{0,1}$

conformal Killing equation on S^2 : $\bar\eth\mathcal{Y}=0=\eth\bar{\mathcal{Y}}$ $\hookrightarrow$ $so(3,1)$ Lorentz algebra

$$\xi = f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth \;,\quad f=\mathcal{T}+\tfrac{1}{2}u\left(\eth\mathcal{Y}+\bar\eth\bar{\mathcal{Y}}\right) ,\quad \mathcal{T}=\mathcal{T}(\zeta,\bar\zeta)$$

integration on S^2 $\quad \langle K^{s,-w-2},\eta^{s,w}\rangle = \frac{1}{4\pi R^2}\int_{S^2}\frac{i\,d\zeta\, i\,d\bar\zeta}{P_s\bar P_s}\,\overline{K^{s,-w-2}}\,\eta^{s,w}$

BMS metric $\Longleftrightarrow$ $\boxed{\text{NP first order}}$

Solution space, free data at $\mathscr{J}^+$: $\boxed{\overset{\circ}{\psi}_2 + \overline{\overset{\circ}{\psi}}_2 ,\quad \overset{\circ}{\psi}_1 ,\quad \overset{\circ}{\sigma}}$ undetermined u-dependence

$\dot{\overset{\circ}{\sigma}}$ news

evolution equations. $\partial_u \overset{\circ}{\psi}_2 = \eth \overset{\circ}{\psi}_3 + \overset{\circ}{\sigma} \overset{\circ}{\psi}_4$, $\partial_u \overset{\circ}{\psi}_1 = \eth \overset{\circ}{\psi}_2 + 2 \overset{\circ}{\sigma} \overset{\circ}{\psi}_3$

constraints $\overset{\circ}{\psi}_2 - \overline{\overset{\circ}{\psi}}_2 = \bar{\eth}^2 \overset{\circ}{\sigma} - \eth^2 \overline{\overset{\circ}{\sigma}} + \dot{\overset{\circ}{\sigma}} \overline{\overset{\circ}{\sigma}} - \overset{\circ}{\sigma} \dot{\overline{\overset{\circ}{\sigma}}}$

$\overset{\circ}{\psi}_3 = - \eth \dot{\overline{\overset{\circ}{\sigma}}}$, $\overset{\circ}{\psi}_4 = - \ddot{\overline{\overset{\circ}{\sigma}}}$

additional data to construct solutions

$$\psi_0 = \sum_{n \geq 0} \psi_0^n (\xi, \bar{\xi}, u_0)\, r^{-5-n}$$

Transformation of (relevant) free data at $\mathcal{J}^+$ $s = (\mathcal{Y}, \mathcal{Y}, \mathcal{T})$, $f = \mathcal{T} + \frac{1}{2}u(\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}})$

$$\delta_s \sigma^0 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}}]\sigma^0 - \eth^2 f$$

$$\delta_s \psi_2^0 = [\quad u \quad u \quad u \quad + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}}]\psi_2^0 + 2\eth f\, \psi_3^0$$

$$\delta_s \psi_1^0 = [\quad u \quad u \quad u \quad + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}]\psi_1^0 + 3\eth f\, \psi_2^0$$

(constraints to be imposed)

broken current algebra

$$\mathcal{J}_s = \frac{i}{R^2}\left[(P_S\bar{P}_S)^{-1} \mathcal{J}^u_s\, d\xi \wedge d\bar{\xi} + P_S^{-1}\mathcal{J}^{\bar{\xi}}_s\, du \wedge d\xi - \bar{P}_S \mathcal{J}^{\xi}_s\, du \wedge d\bar{\xi}\right]$$

$$\delta_{s_1}\mathcal{J}_{s_2} + \boxed{\Theta_{s_2}(\delta_{s_1}X)} \approx -\mathcal{J}_{[s_1,s_2]} + d\, L_{s_1,s_2}$$

non-conservation

$$d\,\mathcal{J}_s + \Theta_s(\delta_{(0,0,1)}X) \approx 0$$

$s_1 : (\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{T}_1) = (0,0,1)$ $\Theta_s(\delta X) \sim \dot{\sigma}^0, \dot{\bar{\sigma}}^0$ vanishes in the absence of news time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{ \Big[\overbrace{\psi^0_2 + \bar{\psi}^0_2}^{\text{BH}} + \overbrace{\sigma^0 \dot{\bar{\sigma}}^0 + \bar{\sigma}^0 \dot{\sigma}^0}^{\text{gravitons}}\Big] f + \Big[\psi^0_1 + \sigma^0 \eth \bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)\Big] \mathcal{Y} + \Big[\bar{\psi}^0_1 + \bar{\sigma}^0 \bar{\eth} \sigma^0 + \frac{1}{2}\bar{\eth}(\sigma^0\bar{\sigma}^0)\Big] \bar{\mathcal{Y}}\Big\}$$

$$\Theta^u_s(\delta X) = \frac{1}{8\pi G}\Big[\dot{\bar{\sigma}}^0 \delta\sigma^0 + \dot{\sigma}^0 \delta\bar{\sigma}^0\Big] f$$

charges $Q_s = \int_{S^2,\, u = cte} \frac{i}{R^2} \frac{d\zeta\, d\bar{\zeta}}{P_s \bar{P}_s} J^u_s$. $\quad \Theta_s[\delta X] = \int_{S^2,\, u = cte} \frac{i}{R^2} \frac{d\zeta\, d\bar{\zeta}}{P_s \bar{P}_s} \Theta^u_s[\delta X]$

algebra $\quad \delta_{s_1} Q_{s_2} + \Theta_{s_2}[\delta_{s_1} X] = -Q_{[s_1, s_2]}$

(non-)conservation of BMS_4 charges $\quad \frac{d}{du} Q_s = -\underbrace{\int_{S^2,\, u = cte} \frac{i}{R^2\, 8\pi G} \frac{d\zeta\, d\bar{\zeta}}{P_s \bar{P}_s} \Big[\dot{\bar{\sigma}}^0 \delta_s \sigma^0 + \dot{\sigma}^0 \delta_s \bar{\sigma}^0\Big]}_{\text{fluxes} \qquad \text{generalizes Bondi mass loss}}$

Coadjoint representation & semi-direct product groups

adjoint $\quad \mathfrak{g}: [e_a, e_b] = f^c{}_{ab}\, e_c \qquad (\mathrm{ad}\, e_a)^b{}_c = f^b{}_{ac} \iff \mathrm{ad}\, e_a(e_b) = f^c{}_{ab}\, e_c$

coadjoint $\quad \mathfrak{g}^*: \langle e_*^b, e_a \rangle = \delta^b_a \qquad (\mathrm{ad}^* e_a) = -(\mathrm{ad}\, e_a)^T \iff \mathrm{ad}^* e_a(e_*^b) = -f^b{}_{ac}\, e_*^c$

group $\quad \mathrm{Ad}_g\, e_a = g\, e_a\, g^{-1}, \qquad \mathrm{Ad}^*_g = g\, e_*^b\, g^{-1}$

semi-direct product $\quad G \ltimes_\sigma A : \quad (f, \alpha)\cdot(g, \beta) = (f\cdot g,\ \alpha + \sigma_f(\beta))$ $\qquad \sigma$: representation

ISO(3), ISO(3,1), $\qquad\qquad A$: abelian ideal

BMS_3, BMS_4 … $\qquad \mathfrak{g} \oplus_\Sigma A : \quad [(X,\alpha),(Y,\beta)] = ([X,Y],\ \Sigma_X\beta - \Sigma_Y\alpha)$

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X,\ \sigma_f\beta - \Sigma_{\mathrm{Ad}_f X}\,\alpha)$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y],\ \Sigma_X\beta - \Sigma_Y\alpha)$$

dual space $\quad \mathfrak{g}^* \oplus A^* \qquad \langle (j,p), (X,\alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle$

terminology $\quad j$: angular momentum $\qquad p$: linear momentum $\qquad$ BMS: add "super"

X : inf. rotation $\qquad \alpha$: inf. translation

ingredients $\quad \times : A \oplus A^* \to \mathfrak{g}^* \quad : \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle$

change in angular momentum due to a translation

$$\sigma^* : G \times A^* \to A^* \quad : \quad \langle \sigma^*_f p, \alpha \rangle = \langle p, \sigma_{f^{-1}} \alpha \rangle$$

coadjoint representation $\quad \mathrm{Ad}^*_{(f,\alpha)} (j,p) = \left(\mathrm{Ad}^*_f j + \alpha \times \sigma^*_f p, \ \sigma^*_f p \right)$

$$\mathrm{ad}^*_{(X,\alpha)} (j,p) = \left(\mathrm{ad}^*_X j + \alpha \times p, \ \Sigma^*_X p \right)$$

Ingredients

(super-)translation $\mathcal{T} : [0,1]$ real

(super-) rotation $\mathcal{Y} : [-1,1]$ $\bar{\eth}\mathcal{Y} = 0$

$\bar{\mathcal{Y}} : [1,1]$ $\eth\bar{\mathcal{Y}} = 0$

(super-) momentum $\mathcal{P} : [0,-3]$ real

(super-) angular momentum $\mathcal{J} : [-1,-3]$ $\mathcal{J} \sim \mathcal{J} + \eth \underset{[-2,-2]}{\mathcal{L}}$,

$\bar{\mathcal{J}} : [1,-3]$ $\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\eth}\underset{[2,-2]}{\bar{\mathcal{L}}}$,

bms_4 algebra $\quad [(\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{J}_1), (\mathcal{Y}_2, \bar{\mathcal{Y}}_2, \mathcal{J}_2)] = (\hat{\mathcal{Y}}, \hat{\bar{\mathcal{Y}}}, \hat{\mathcal{J}})$

$$\hat{\mathcal{Y}} = \mathcal{Y}_1 \eth \mathcal{Y}_2 - \mathcal{Y}_2 \eth \mathcal{Y}_1 \qquad \hat{\mathcal{J}} = \mathcal{Y}_1 \eth \mathcal{J}_2 - \frac{1}{2} \eth \mathcal{Y}_1 \mathcal{J}_2 - (1 \leftrightarrow 2) + c.c.$$

subalgebra $\mathfrak{g}$ $\quad (\mathcal{Y}, \bar{\mathcal{Y}}, 0)$

(Lorentz, Witt $\oplus$ Witt)

representation of $\mathfrak{g}$ on $\eta^{s,w}$

$$\mathcal{Y} \cdot \eta^{s,w} = \mathcal{Y} \eth \eta^{s,w} + \frac{s-w}{2} \eth \mathcal{Y} \, \eta^{s,w}$$

$$\bar{\mathcal{Y}} \cdot \eta^{s,w} = \bar{\mathcal{Y}} \bar{\eth} \eta^{s,w} - \frac{s+w}{2} \bar{\eth} \bar{\mathcal{Y}} \, \eta^{s,w}$$

$$\Sigma_x \alpha = (\mathcal{Y}, \bar{\mathcal{Y}}) \cdot \mathcal{T}^{[0,1]}$$

action of inf rotation on translations

$\mathfrak{bms}_4^*$ dual space $([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P})$ $([\tilde{\mathcal{J}}], [\hat{\tilde{\mathcal{J}}}], \tilde{\mathcal{P}})$

$[0,-2]$

pairing $\langle ([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}); (\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T}) \rangle = \int_S d\mu\, [\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}]$, $d\mu(\zeta,\bar{\zeta}) = \frac{iC}{PP} d\zeta \wedge d\bar{\zeta}$

pairing annihilates total $\eth, \bar{\eth}$ derivatives
non-degenerate $\rightarrow$ integrations by parts

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{=\mathrm{ad}^*_{\bar{\mathcal{Y}}}\mathcal{J} \sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha \times \mathcal{P}},$$

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\mathcal{P} + \text{c.c.}}_{\mathcal{E}^*_X \mathcal{P}}$$

work out formulas for the group

Expansions : spin weighted spherical harmonics: ${}_sZ_{jm}$ unnormalized ${}_sY_{jm}$ normalized

Gelfand, Minlos, Shapiro (1958); Wu & Yang, Nucl. Phys. B (1976)
Newman, Penrose, JMP (1966); Thorne, Rev. Mod. Phys (1980)

conformal Killing eq. on S^2 $\quad \bar{\eth} \mathcal{Y}^{[-1,1]} = 0 = \eth \bar{\mathcal{Y}}^{[1,-1]}$

$$\mathcal{Y}_m = -R\sqrt{2}\; {}_{-1}Z_{1,m} \qquad m = -1,0,1 \qquad \mathcal{Y} = \sum_{m=-1}^{1} y_m \mathcal{Y}_m$$

$$T_{jm} = {}_0Z_{j,m} \qquad T = \sum_{j,|m|\leq j} t_{j,m} T_{j,m}, \qquad \bar{t}_{jm} = (-1)^m t_{j,-m}$$

dual basis $\quad y^m_* = \dfrac{-6}{R\sqrt{2}(1+m)!(1-m)!}\; {}_{-1}Z_{1,m} \qquad J = \sum_{m=-1}^{1} j_m\, y^m_*$

$$T^{j,m}_* = \frac{(2j+1)!(2j)!}{j!\,j!\,(j+m)!(j-m)!}\; {}_0Z_{j,m} \qquad P = \sum_{j,|m|\leq j} p_{j,m} T^{j,m}_*, \qquad \bar{p}_{jm} = (-1)^m p_{j,-m}$$

$$: \quad [\mathcal{Y}_m, \mathcal{Y}_n] = (m-n)\,\mathcal{Y}_{m+n}$$

$\to$ all other structure constants can be worked out explicitly (ugly)

Remark   Penrose & Rindler Vol I, section 4.15

$w \geq |s|$ $\eth^{w+s+1}\,\eta^{s,w}$ $\bar\eth^{w+s+1}\,\eta^{s,w}$

$[w+1, s-1]$ $[-w-1, -s-1]$

definite boost weight

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

j \ s	$\dots$	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\dots$
0							1						
$\frac{1}{2}$						2		2					
1		$\overleftarrow{\eth'}$			3		3		3		$\overrightarrow{\eth}$		
$\frac{3}{2}$				4		4		4		4			
2			5		5		5		5		5		
$\frac{5}{2}$		6		6		6		6		6		6	
$\vdots$	$\dots$												$\ddots$

(4.15.60)

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator $\eth$ carries us a step of one s-unit to the right and $\eth'$ one s-unit to the left. (From our earlier discussion, the j-value is not affected by $\eth$ or $\eth'$.) Whenever such a step carries us off the array, the result of the operator $\eth$ or $\eth'$ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$\bar\eth\eta = 0 \iff \eth^3\eta = 0$

$\eth\bar\eta = 0 \iff \bar\eth^3\bar\eta = 0$ same solutions

dual situation $w \leq -|s|-2$

$\eth^{s-w-1}\,\kappa^{w+1,s-1}$ $\bar\eth^{-s-w-1}\,\kappa^{-w-1,-s-1}$

$[s,w]$ $[s,w]$ definite boost weight

$[2,-2]$ $[-2,0]$

$\tilde J \sim \tilde J + \bar\eth\,\bar L \iff \bar{\tilde J} \sim \bar{\tilde J} + \eth^3 M$

same equivalence classes

$j \leq w$: finite dim. repres of Lorentz, "heads"

$j > w$: ∞ dim, "tails"

Remark (ii)   reduction to Poincaré

$\eth^2 J = 0 = \bar\eth^2 J$ $P \sim P + \eth^2 N + \bar\eth^2 \bar N$

$[-2,1]$ $[2,-1]$

$\boxed{\text{non-radiative space-times (no news)}}$ $\sigma^0 = \sigma^0(\zeta, \bar\zeta, \cancel{u})$ $\left(\Rightarrow \dot\sigma^0 = 0 = \psi_3^0 = \psi_4^0, \quad \mathcal{O}_s[\delta X] = 0 \right)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u = 0$ $\quad \mathcal{P} = -\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)$ $\quad \bar{\mathcal{J}} = -\frac{1}{2G}\big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar\sigma^0 + \frac{1}{2}\eth(\sigma^0\bar\sigma^0)}_{\psi^0_{1\bar{\mathcal{J}}}}\big)$

super-momentum = Bondi mass aspect

~~super~~- angular momentum = Bondi angular momentum aspect

(pre)-momentum map: $\mathcal{F}$. algebra of non-radiative free data

$\mathfrak{bms}_4$ representation δ_s , $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathcal{F} \to \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)\right) = \mathcal{P}, \quad \mu\left(-\frac{1}{2G}\psi^0_{1\bar{\mathcal{J}}}\right) = [\bar{\mathcal{J}}], \quad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

transformation laws at $\mu = 0$ $\qquad \delta_S(\psi_2^0 + \bar\psi_2^0) = (yj + \bar y\bar j + \frac{3}{2} jy + \frac{3}{2}\bar j\bar y)(\psi_2^0 + \bar\psi_2^0)$

$$\delta_S \psi_1^0 = [yj + \bar y\bar j + 2jy + \bar j\bar y]\psi_1^0 + \frac{1}{2}\bar J j(\psi_2^0 + \bar\psi_2^0 + \bar j^2\cancel{\tau^0} - j^2\cancel{\bar\tau^0})$$
$$+ \frac{3}{2} j\bar J(\psi_2^0 + \bar\psi_2^0 + \bar j^2\cancel{\tau^0} - j^2\cancel{\bar\tau^0})$$

$$\delta_S \psi^0_{1\bar j} = [yj + \bar y\bar j + 2jy + \bar j\bar y]\psi^0_{1\bar j} + \frac{1}{2}\bar J j(\psi_2^0 + \bar\psi_2^0) + \frac{3}{2} j\bar J(\psi_2^0 + \bar\psi_2^0)$$

$$+ \frac{1}{2}\bar j(\bar J\bar j j\tau^0 - \bar j\bar J j\tau^0 + 3jJ\bar j\tau^0 - 3\bar j jJ\tau^0 - \frac{3}{2}J\tau^0)\ \boxed{-\frac{1}{2}j^3(\bar J\bar\tau^0)}$$

$\boxed{\text{trivial!}}$

Remark: electric case $\bar j^2\tau_e^0 = j^2\bar\tau_e^0 \Leftrightarrow \tau_e^0 = j^2\chi_e$

$$\delta_S\chi_e = [yj + \bar y\bar j - \frac{1}{2}jy - \frac{1}{2}\bar j\bar y] - J$$

simplified pre-momentum map $\mu' : \mathcal{F}_e \longrightarrow \mathfrak{dms}_4^*$

$$\mu'[-\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)] = \mathcal{T}, \quad \mu'[-\frac{1}{2G}\psi_1^0] = [\mathcal{J}], \quad \mu'\circ\delta_S = \mathrm{ad}^*_S\circ\mu'$$

Further developments

- extended BMS_4 : use Weyl transformation $d\tilde{s}^2 = -dz\,d\bar{z}$

  2-punctured Riemann sphere

  $\bar{\partial} y = 0 = \partial \bar{y} \qquad y = y(z), \quad \bar{y} = \bar{y}(\bar{z})$

  as in CFT 2 copies of Witt algebra instead of $so(3,1)$

  used in celestial holography

- generalized BMS_4 : remove chirality condition

  $\mathrm{Diff}(S^2)$ instead of $so(3,1)$